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On Some Dynamical Properties of Discontinuous Dynamical Systems

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Abstract: In this work we are concerned with the discontinuous dynamical system representing the problem of the logistic retarded functional equation with delay $r > 0$. The existence and uniqueness of the solution will be proved. The local stability at the equilibrium points will be studied. The bifurcation analysis and chaos will be discussed.

Keywords: Logistic functional equation, existence, uniqueness, equilibrium points, local stability, bifurcation, chaos.

1 Introduction

Consider the problem of retarded functional equation,

$$x(t) = f(x(t-r)), \qquad t \in (0,T] \tag{1}$$

$$x(t) = x_0, \qquad t \le 0. \tag{2}$$

Let $t \in (0,r]$, then $t - r \in (-r,0]$ and the solution of (1)–(2) is given by

$$x(t) = x_r(t) = f(x_0), \qquad t \in (0,r].$$

For $t \in (r,2r]$, we find that $t - r \in (0,r]$ and the solution of (1)–(2) is given by

$$x(t) = x_{2r}(t) = f(x_r(t)) = f^2(x_0), \quad t \in (r,2r].$$

Repeating the process we can deduce that the solution of the problem (1)–(2) is given by

$$x(t) = x_{nr}(t) = f^n(x_0), \qquad t \in ((n-1)r, nr],$$

which is continuous on each subinterval $((k-1)r, kr),\ k = 1,2,\ldots,n$, but

$$\lim_{t \to kr^+} x_{(k+1)r}(t) = f^{k+1}(x_0) \neq x_{kr}(t),$$

which implies that the solution of the problem (1)–(2) is discontinuous (sectionally continuous) on $(0,T]$.

So, we can give the following definition,

Definition 1 *The discontinuous dynamical system is a problem of retarded functional equation,*

$$x(t) = f(t, x(t-r_1), x(t-r_2), \ldots, x(t-r_n)), \quad t \in (0,T],$$
$$x(t) = x_0, \quad t \le 0 \tag{3}$$

Definition 2 *The equilibrium points of the discontinuous dynamical system* (3) *is the solutions of the equation,*

$$x(t) = f(t, x, x, \ldots, x).$$

Consider now the discontinuous dynamical system of the Logistic retarded functional equation with delay $r > 0$,

$$x(t) = \rho x(t-r)[1 - x(t-r)], \quad t \in (0,T] \tag{4}$$

$$x(t) = x_0, \qquad t \le 0, \tag{5}$$

we study here the existence of a unique continuously dependent solution $x \in L^1$ of theproblem (4)−(5). The asymptotic stability (see [1]− [7]) at the equilibrium points will be studied. To study bifurcation and chaos, we take firstly $r = 1$ and we compare the results with the results of the discrete dynamical system Logistic difference equations,

$$x_n = \rho x_{n-1}[1 - x_{n-1}], \quad n = 1,2,\ldots . \tag{6}$$

Secondly, we take some other values of r and T will be studied as examples.

2 Existence and Uniqueness

Let $L^1 = L^1[0,T],\ T < \infty$ be the class of Lebesgue integrable functions on $[0, T]$ with norm

$$\|f\| = \int_0^T |f(t)|\,dt, \quad f \in L^1.$$

Let $D=\{x \in R: 0 \le x(t) \le 1,\ t \in (0,T]$ and $x(t)=x_0,\ t \le 0\}$.

Definition 3 *By a solution of the problem* (4)−(5) *we mean a function* $x \in L^1$ *satisfying the problem* (4)−(5).

Theorem 1 *The problem* (4) − (5) *has a unique solution* $x \in L^1$.

Proof. Define, on D, the operator $F: L^1 \to L^1$ by

$$Fx(t) = \rho x(t-r)[1-x(t-r)].$$

The operator *F* makes sense, indeed for $x \in D$ we have

$$|Fx(t)| \le \rho\, |x(t-r)|$$

and

$$\| Fx(t) \| \le r \rho x_0 + \rho \| x \|.$$

Now for $x, y \in D$, we can obtain

$$|Fx - Fy| \le |\rho x(t-r)(1-x(t-r)) - \rho y(t-r)(1-y(t-r))| \le$$
$$\le \rho |x(t-r) - y(t-r)|,$$

which implies that

$$\|Fx - Fy\| \le \rho \int_0^T |x(t-r) - y(t-r)| dt =$$
$$= \rho [\int_0^r |x(t-r)-y(t-r)| dt + \int_r^T |x(t-r)-y(t-r)| dt] =$$
$$= \rho [\int_r^T |x(t-r)-y(t-r)| dt] \le \rho \|x-y\|.$$

If $\rho < 1$, we deduce that

$$\| Fx - Fy \| \le \| x - y \|$$

and then the problem (4)–(5) has, on *D*, a unique solution $x \in L^1$. □

3 Continuous dependence on initial conditions

Theorem 2 *If* $\rho < 1$. *Then the solution of the discontinuous dynamical system rep-resents the problem of the logistic retarded functional equation with delay* (4)−(5) *is continuously dependent on the initial data in the sense that,*

$$|x_0 - x_0^*| \le \delta \quad \Rightarrow \quad \|x - x^*\| \le \varepsilon$$

where x^* is the solution of the problem,

$$x(t) = \rho x(t-r)[1-x(t-r)], \quad t \in (0,T],$$
$$x(t) = x_0^*, \qquad t \le 0, \qquad (7)$$

Proof. Let $x(t)$ and $x^*(t)$ be the solution of the two problems (4)–(5) and (4)–(7) respectively, then,

$$|x(t) - x^*(t)| \le \rho |x(t-r) - x^*(t-r)|,$$

which implies that

$$\|x(t) - x^*(t)\| \le \rho \int_0^T |x(t-r) - x^*(t-r)| dt =$$
$$= \rho [\int_0^r |x(t-r) - x^*(t-r)| dt + \int_r^T |x(t-r) - x^*(t-r)| dt] \le$$
$$\le \rho [|x_0 - x_0^*| \int_0^r dt + \|x - x^*\|] = \rho r |x_0 - x_0^*| + \rho \|x - x^*\|,$$

and

$$\|x - x^*\| \le \frac{\rho r}{1-\rho} |x_0 - x_0^*|,$$

which proves that

$$|x_0 - x_0^*| \le \delta \quad \Rightarrow \quad \|x - x^*\| \le \varepsilon = \frac{\rho r}{1-\rho}\delta,$$

and the theorem is proved. □

4 Equilibrium Points and their asymptotic stability

The equilibrium points of (4) are the solution of the equation

$$\rho x_{eq}(1 - x_{eq}) = x_{eq},$$

which are

$$(x_{eq})_1 = 0,$$
$$(x_{eq})_2 = 1 - \frac{1}{\rho}.$$

The equilibrium point of (4) is locally asymptotically sable if all the roots λ of the equation,

$$\lambda^r = \rho(1 - 2x_{eq}), \qquad (8)$$

satisfy $|\lambda| < 1$ (see [8]).

Then the equilibrium point $x_{eq} = 0$ is locally asymptotically sable if $\rho < 1$, while the second equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$ is locally asymptotically sable if all the roots λ of the equation,

$$\lambda^r = \rho(1 - 2(1 - \frac{1}{\rho})) = 2 - \rho, \qquad (9)$$

satisfy $|\lambda| < 1$.

The equilibrium point $x_{eq} = 0$ is locally asymptotically sable if $\rho < 1$, which is the same as in the discrete case (6). Also, when *r* = 1, we deduce that the equilibrium point $x_{eq} = 1 - \frac{1}{\rho}$, $\rho > 1$ is locally asymptotically sable if $1 < \rho < 3$, which is the same as in the discrete case (6).

In studying (4) − (5) it may be useful to study the difference equation (6).

5 Bifurcation and Chaos

In this section, some numerical simulations results are presented to show that dynamics behaviors of the discontinuous dynamical system (4)–(5) change for different values of r and T. To do this, we will use the bifurcation diagrams as follow:-

Example 1

1. we take $r = 1$ and $t \in [0, 50]$, in this case, we get the same behavior as in the discrete case (Figure 1).
2. we take $r = 2$ and $t \in [0, 50]$ (Figure 2).
3. we take $r = 1.75$ and $t \in [0, 50]$ (Figure 3).

Example 2

1. we take $r = 0.1$ and $t \in [0, 5]$ (Figure 4).
2. we take $r = 0.2$ and $t \in [0, 5]$ (Figure 5).
3. we take $r = 0.3$ and $t \in [0, 5]$ (Figure 6).

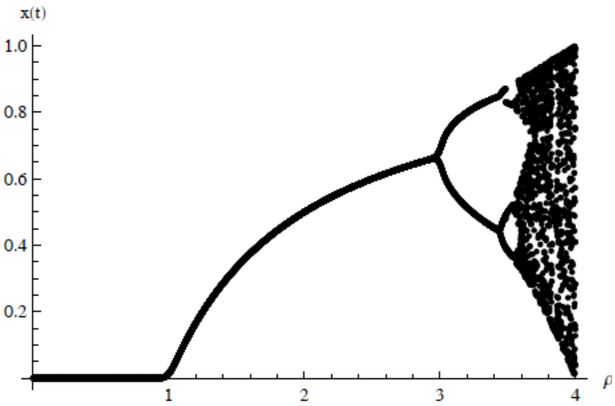

Figure 1: Bifurcation diagram of map (4)–(5) with respect to ρ, $r = 1$ and $t \in [0, 50]$.

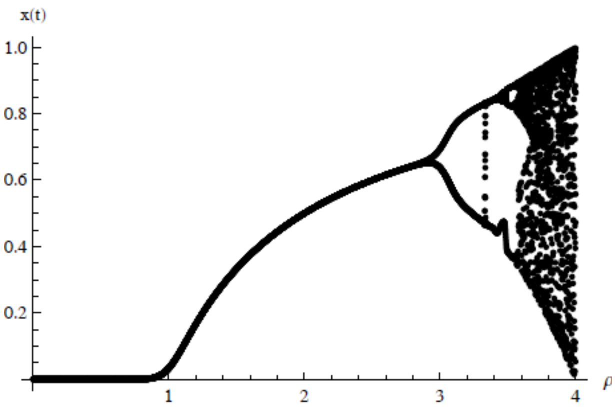

Figure 2: Bifurcation diagram of map (4)–(5) with respect to ρ, $r = 2$ and $t \in [0, 50]$.

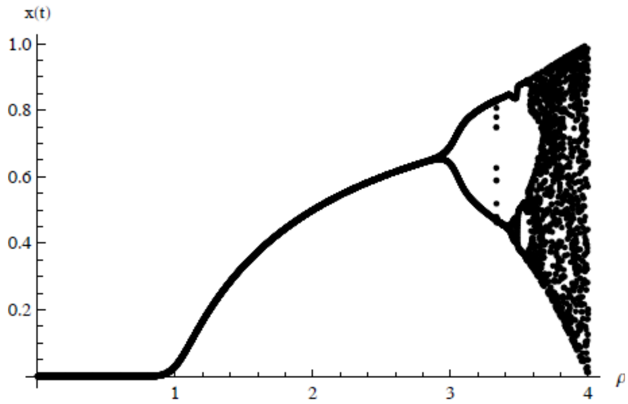

Figure 3: Bifurcation diagram of map (4)–(5) with respect to ρ, $r = 1.75$ and $t \in [0, 50]$.

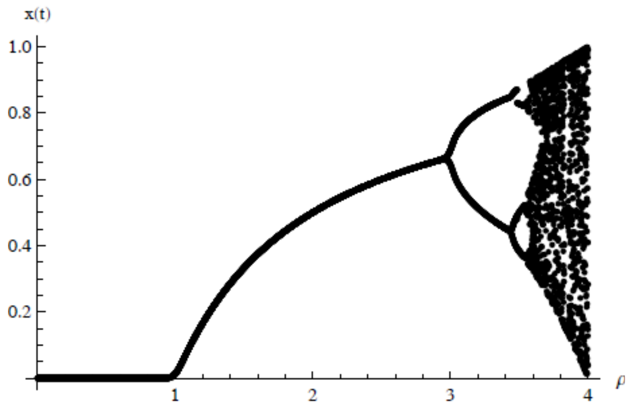

Figure 4: Bifurcation diagram of map (4)–(5) with respect to ρ, $r = 0.1$ and $t \in [0, 5]$.

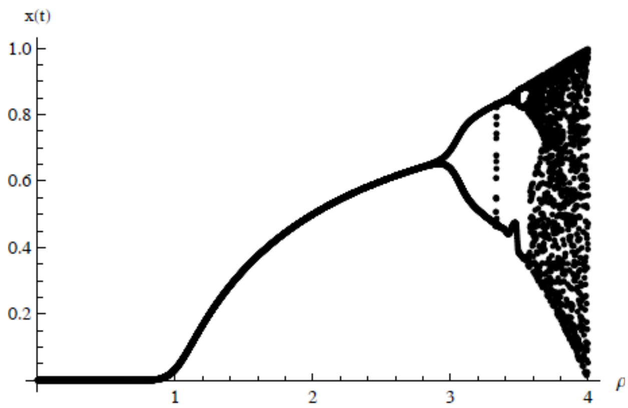

Figure 5: Bifurcation diagram of map (4)–(5) with respect to ρ, $r = 0.2$ and $t \in [0, 5]$.

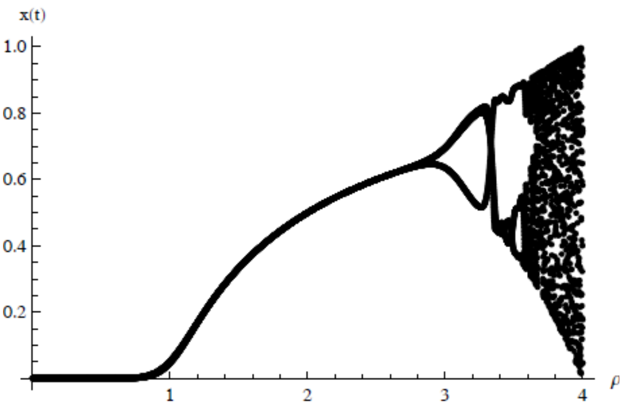

Figure 6: Bifurcation diagram of map (4)−(5) with respect to ρ , r = 0.3 and $t \in$ [0,5].

From Figures (1−6) we deduce that the change of r and T effect of stability of the Logistic equation model, occurs of a bifurcation point, parameter sets for which a periodic behavior occur and parameter sets for which a chaotic behavior occur.

6 Conclusions

The discrete dynamical system of the Logistic equation model describes the dynamical properties for the case r = 1 and the time is discrete t = 1,2,… .

On the other hand, the discontinuous dynamical system of the Logistic equation model describes the dynamical properties for different values of the delayed parameter $r \in R^{+}$ and the time $t \in [0, T]$ is continuous.
Figures (1),(4) agrees with the results of the asymptotic stability, this confirm that our numerics are correct. Also from figures (1),(4) and (2),(5), it locks like that there is a scale that gives identical chaos behavior.
This shows the richness of the models of discontinuous dynamical systems.

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